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The Derivative of a Function of a Surface.

BY CHARLES A. FISCHER.

Introduction.

In a paper published in the *Atti della R. Academia dei Lincei,*[†] Volterra has defined the derivative of a function of a curve, and has proved that if the derivative is continuous and approached uniformly, the first variation of the function is equal to the integral of the product of this derivative and the first variation of the dependent variable y, taken between limits equal to the values of the independent variable x at the end-points of the given curve. In a paper entitled "A Generalization of Volterra's Derivative of a Function of a Curve,"[‡] I have modified the definition of this derivative to make it applicable to a restricted set of curves defined by means of some ordinary differential equations, and applied some resulting theorems to the Lagrange problem of the calculus of variations.

In the present paper I have given a definition of the derivative of a function of a surface analogous to Volterra's derivative of a function of a curve, and have proved substantially that if the derivative is continuous and approached uniformly with any finite order, then the first variation of the given function is equal to the double integral of the product of the derivative of the function and the first variation of the dependent variable z, the integration taking place over the projection of the given surface on the x, y-plane. I have also generalized this definition and theorem to make them applicable to a restricted set of surfaces, in a manner somewhat similar to that employed in my paper already referred to.

The first section contains the definition of the derivative of a function of a surface, the theorem mentioned above, and a proof that the derivative must vanish at every point of a surface which minimizes the given function. In § 2

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[*] Presented to the American Mathematical Society, April 26, 1913.

[†] Volterra, *Atti della R. Academia dei Lincei,* Ser. IV, Rend. III_2, p. 97.

[‡] Fischer, AMERICAN JOURNAL OF MATHEMATICS, Vol. XXXV (1913), No. 4, p. 369.

the given function is taken as a double integral. Its derivative is found, and an application is made to the theory of maxima and minima of double integrals. In the remainder of the paper only those surfaces which give previously assigned values to a set of functions $M_1(S)$, $M_2(S)$, ..., $M_m(S)$ are considered admissible. In § 3 it is proved that in the neighborhood of any admissible surface for which a certain determinant does not vanish, there are other admissible surfaces of the kind needed in finding the derivative of another function $L(S)$ relative to this restricted set of surfaces. In § 4 this derivative is defined and the theorems of the first section are proved to hold for the special set of surfaces also. In the last section all of the functions of surfaces considered are taken to be double integrals, and the results obtained in § 4 are applied to the isoperimetric problem of the theory of maxima and minima of double integrals.

§ 1. *Definition of the Derivative and Volterra's Theorem.*

All of the functions of x and y considered in this paper will be defined over a region R in the x, y-plane, and will be assumed to be of class $C^{(r)}$,* where r is an arbitrary positive integer. In the first part of this section R will be taken as a rectangle defined by the inequalities $a \leqq x \leqq b$ and $c \leqq y \leqq d$, but later this restriction will be removed and R will be an arbitrary closed region bounded by a curve of finite length.

The surfaces S and S_ϵ will be frequently mentioned. They will be defined by the equations

$$\begin{aligned} S\ : &\quad z = z(x, y), \\ S_\epsilon : &\quad z = z(x, y) + \eta(x, y), \end{aligned}$$

where the function $\eta(x, y)$ is assumed to have the following properties: It vanishes for all values of x and y outside of the region defined by the inequalities $x_0 - \varepsilon < x < x_0 + \varepsilon$ and $y_0 - \varepsilon < y < y_0 + \varepsilon$, or as much of that region as is inside of R if the point (x_0, y_0) happens to be on the boundary of R. Inside this region it has a permanent sign and is not identically zero. The absolute value of each of its partial derivatives up to and including those of order r is everywhere less than ε, the function itself being considered as the partial derivative of order zero.

The derivative of the function $L(S)$ of the surface S can now be defined as follows: *If the limit*

$$L'(S;\, x_0, y_0) = \lim_{\epsilon=0} \frac{L(S_\epsilon) - L(S)}{\sigma}$$

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* Bolza, "Vorlesungen über Variationsrechnung," p. 14.

exists uniquely, where σ *is defined by the equation*

$$\sigma=\int_{x_0-\epsilon}^{x_0+\epsilon}\int_{y_0-\epsilon}^{y_0+\epsilon}\eta(x,\, y)\,dy\,dx, \tag{1}$$

then this limit is said to be the derivative of the function $L(S)$ *at the point* (x_0, y_0), *and it is said to be approached with order* r.

It will be assumed that if $\bar{S}$ is any surface in a fixed neighborhood of order r of the surface S, the function $L(S)$ has the following property: For every $\delta>0$ there exists an $\varepsilon>0$ independent of $\bar{S}$ and (x_0, y_0), such that the inequality

$$\left|\frac{L(\bar{S}_\epsilon)-L(\bar{S})}{\sigma}-L'(\bar{S};\, x_0, y_0)\right|<\delta$$

is always satisfied. That is to say, the derivative is approached uniformly with order r in the neighborhood of the surface S.

It will be assumed that for every $\delta>0$ there exists an $\varepsilon>0$ independent of (x_0, y_0) such that if $\bar{S}$ is any surface in the neighborhood $(\varepsilon)_r$ of S, then the inequality

$$|L'(\bar{S};\, x_0, y_0)-L'(S;\, x_0, y_0)|<\delta$$

is satisfied. That is to say, $L'(S;\, x, y)$ is continuous with order r in the argument S. It will also be assumed that $L'(S;\, x, y)$ is a continuous function of the variables x and y, and since the region R is closed, it is uniformly continuous.

A family of surfaces will now be considered which is defined by the equation

$$S_\alpha: \qquad z=z(x,\, y)+\omega(x,\, y,\, \alpha),$$

where the function $\omega(x, y, \alpha)$ has the following properties: All of its partial derivatives with respect to x and y vanish identically in α at every point of the boundary of R. They also approach zero uniformly in x and y when α approaches zero, and vanish when $\alpha=0$. If a numerically small constant value is given to α, the function $\omega(x, y, \alpha)$ has a permanent sign. The derivative $\omega_\alpha(x, y, 0)$ exists, is continuous and approached uniformly.

The extension of Volterra's theorem, mentioned in the introduction, to functions of surfaces will be stated as follows: *If the function* $L(S)$ *and the surface* S_α *satisfy the assumptions given above, then the derivative of* $L(S_\alpha)$ *with respect to the parameter* α *is given by the equation*

$$\left.\frac{dL(S_\alpha)}{d\alpha}\right|_{\alpha=0}=\iint_R L'(S;\, x,\, y)\,\omega_\alpha(x,\, y,\, 0)\,dy\,dx.$$

Select the constant ε arbitrarily. Then divide the region R into n^2 equal rectangles, calling their vertices (x_i, y_j), where i and j have the range 0, 1, 2,

$\ldots\ldots, n$, and taking n so large that the dimensions of each rectangle are less than ε. Evidently $x_0=a$, $x_n=b$, $y_0=c$, and $y_n=d$. Now choose another positive constant h satisfying the equation $nh(b-a+d-c)=\varepsilon$. A set of functions $\theta_1(x), \theta_2(x), \ldots\ldots, \theta_n(x)$ will then be selected, satisfying the conditions

$$\begin{array}{rll} \theta_i(x)=0, & (a\leqq x_i\leqq x_{i-1} \text{ and } x_i\leqq x\leqq b), & (i=1, 2, \ldots\ldots, n), \\ 0<\theta_i(x)<1, & (x_{i-1}<x<x_{i-1}+h \text{ and } x_i-h<x<x_i), & \\ \theta_i(x)=1, & (x_{i-1}+h\leqq x\leqq x_i-h). & \end{array}$$

Another set of functions $\phi_1(y), \phi_2(y), \ldots\ldots, \phi_n(y)$ will also be chosen which satisfy the analogous conditions. The surfaces S_{ij} will then be defined by the equations

$$S_{ij}: \qquad z=z(x, y)+\left[\sum_{k=1}^{i-1}\sum_{l=1}^{n}\theta_k(x)\phi_l(y)+\sum_{l=1}^{j}\theta_i(x)\phi_l(y)\right]\omega(x, y, \alpha),$$
$$(i, j=1, 2, \ldots\ldots, n).$$

For convenience in notation $S_{i-1\,n}$ will sometimes be written S_{i0}.

After ε and n have been chosen, it is possible to find a positive constant α_0 such that if $|\alpha|<\alpha_0$ the surfaces S_{ij} will all be in the neighborhood $(\varepsilon)_r$ of S. Then S_{ij} is the kind of varied surface used in forming the derivative

$$L'(S_{i\,j-1}; x_i, y_j),$$

and since it is assumed to be approached uniformly, the following equation is satisfied:

$$\frac{L(S_{nn})-L(S)}{\alpha}=$$
$$\sum_{i, j=1}^{n}[L'(S_{i\,j-1}; x_i, y_j)+\xi_{ij}]\textstyle\int_{x_{i-1}}^{x_i}\int_{y_{j-1}}^{y_j}\theta_i(x)\phi_j(y)\,\dfrac{\omega(x, y, \alpha)}{\alpha}\,dydx, \qquad (2)$$

where ξ, the upper bound of $|\xi_{ij}|$, approaches zero with ε. The derivatives $L'(S_{i\,j-1}; x_i, y_j)$ may be taken at any points of the proper rectangles instead of at the corners, if desired. Since it is assumed to be continuous in the argument S, $L'(S_{i\,j-1}; x_i\, y_j)$ can be replaced by $L'(S; x_i, y_j)$. This changes the values of the quantities ξ_{ij} but does not affect the property of their upper bound mentioned above. The double integral in the right member of equation (2) can be written

$$\int_{x_{i-1}}^{x_i}\int_{y_{j-1}}^{y_j}\frac{\omega(x, y, \alpha)}{\alpha}\,dydx-\int_{x_{i-1}}^{x_i}\int_{y_{j-1}}^{y_j}[1-\theta_i(x)\phi_j(y)]\,\frac{\omega(x, y, \alpha)}{\alpha}\,dydx$$
$$=\int_{x_{i-1}}^{x_i}\int_{y_{j-1}}^{y_j}\frac{\omega(x, y, \alpha)}{\alpha}\,dydx+2\lambda hN\frac{b-a+d-c}{n}, \qquad (3)$$

where $-1<\lambda<1$, and

$$N > \left|\frac{\omega(x, y, \alpha)}{\alpha}\right|.$$

The uniform approach to $\omega_\alpha(x, y, 0)$ implies the existence of such a constant N for α_0 sufficiently small. If a constant M is chosen, satisfying the inequalities

$$M > |L'(S; x_i, y_j) + \xi_{ij}|, \qquad (i, j = 1, 2, \ldots, n),$$

the absolute value of the contribution of the last term of the right member of equation (3) to the value of the right member of equation (2) will be less than the quantity $2nhMN(b-a+d-c) = 2MN\varepsilon$, which approaches zero with ε. The absolute value of the sum of the terms involving ξ_{ij} is less than $\xi N(b-a)(d-c)$, which also approaches zero with ε. The mean-value theorem implies that the equation

$$\int_{x_{i-1}}^{x_i}\int_{y_{j-1}}^{y_j} \frac{\omega(x, y, \alpha)}{\alpha}\, dy dx = \frac{\omega(x'_i, y'_j, \alpha)}{\alpha}(x_i - x_{i-1})(y_j - y_{j-1})$$

is satisfied, where (x'_i, y'_j) is a point in the rectangle over which the integration takes place, and in accordance with the remark following equation (2) $L'(S; x'_i, y'_j)$ can be substituted for $L'(S; x_i, y_j)$ in that equation. It follows directly that the equation

$$\lim_{\epsilon, \alpha=0} \frac{L(S_{nn}) - L(S)}{\alpha} = \int_a^b \int_c^d L'(S; x, y)\,\omega_\alpha(x, y, 0)\, dy dx \tag{4}$$

is satisfied.

It remains to be proved that

$$\lim_{\epsilon, \alpha=0} \frac{L(S_\alpha) - L(S_{nn})}{\alpha} = 0. \tag{5}$$

By definition S_{nn} coincides with S_α, excepting over $n+1$ rectangles in the x, y-plane whose dimensions are $b-a$ and either h or $2h$, and $n+1$ other rectangles whose dimensions are either h or $2h$ and $d-c$; and wherever the surfaces meet, the contact is of order r. The overlapping of the two sets of rectangles has no effect on the work here. The following lemma is needed to establish equation (5):

If the surface S_h is given by the equation

$$S_h: \qquad z = z(x, y) + \eta(x, y, \alpha),$$

where the function $\eta(x, y, \alpha)$ vanishes identically over all of the region R excepting the rectangle $(a < x < b;\ y_0 - h < y < y_0 + h)$ where it has a permanent sign when α is kept constant, if there is a finite constant N satisfying the inequality

$$\left|\frac{\eta(x, y, \alpha)}{\alpha}\right| < N,$$

and all of its partial derivatives with respect to x and y approach zero uniformly with α, then, for sufficiently small values of h and $|\alpha|$, the inequality

$$\left|\frac{L(S_h)-L(S)}{\alpha}\right| < 4hMN(b-a) \tag{6}$$

is satisfied, where M is any constant greater than the upper bound of $|L'(S;\, x, y)|$.

This can be proved as follows: Divide the interval $[a, b]$ into n equal parts, calling the points of division $x_1, x_2, \ldots, x_{n-1}$. For convenience in notation let $x_0=a$ and $x_m=b$. Then a set of functions $\psi_1(x), \psi_2(x), \ldots, \psi_{n-1}(x)$ can be found which satisfy the conditions

$$\begin{aligned} \psi_i(x) &= 1, \quad (a \leqq x \leqq x_i), \qquad (i=1, 2, \ldots, n-1), \\ 0 < \psi_i(x) &< 1, \quad (x_i < x < x_{i+1}), \\ \psi_i(x) &= 0, \quad (x_{i+1} \leqq x \leqq b). \end{aligned}$$

Then the surfaces $S_1, S_2, \ldots, S_{n-1}$ will be defined by the equations

$$S_i\colon \qquad z = z(x, y) + \psi_i(x)\eta(x, y, \alpha), \qquad (i=1, 2, \ldots, n-1),$$

and for convenience in notation let $S_0=S$ and $S_n=S_h$. It follows from the definition of the derivative of a surface that

$$L'(S_{i-1};\, x_i, y_0) = \lim_{\substack{h,\, \alpha=0 \\ n=\infty}} \frac{L(S_i)-L(S_{i-1})}{\int_{x_{i-1}}^{x_{i+1}}\int_{y_0-h}^{y_0+h}[\psi_i(x)-\psi_{i-1}(x)]\eta(x, y, \alpha)\,dy\,dx}.$$

Since this derivative is continuous and approached uniformly, the equation

$$\frac{L(S_h)-L(S)}{\alpha} = \sum_{i=1}^{n}[L'(S;\, x_i, y_0)+\xi_i]\int_{x_{i-1}}^{x_{i+1}}\int_{y_0-h}^{y_0+h}[\psi_i(x)-\psi_{i-1}(x)]\,\frac{\eta}{\alpha}\,dy\,dx$$

is satisfied, where the upper bound of $|\xi_i|$ approaches zero with h and α. By definition $|\psi_i(x)-\psi_{i-1}(x)| \leqq 1$, and for sufficiently small values of h and α

$$|L'(S;\, x_i, y_0)+\xi_i| < M, \qquad (i=1, 2, \ldots, n),$$

and

$$\left|\frac{\eta(x, y, \alpha)}{\alpha}\right| < N.$$

The sum of the areas over which the integration takes place is $4h(b-a)$. If their upper bounds are substituted for the respective functions in the last equation, it establishes the inequality (6).

If the equation of S_h is now taken to be

$$z = z(x, y) + \sum_{i=1}^{n}\theta_i(x)\omega(x, y, \alpha),$$

and the surface S_{nn} is taken instead of S, the function $\eta(x, y, \alpha)$ becomes

$$\eta(x, y, \alpha)=\left[1-\sum_{j=1}^{n}\phi_j(y)\right]\sum_{i=1}^{n}\theta_i(x)\omega(x, y, \alpha),$$

which vanishes excepting in the $n+1$ rectangles $(a<x<b;\ y_j-h<y<y_j+h)$, $(j=0, 1, \ldots, n)$. Since the rest of the hypothesis of the lemma is satisfied, if it is applied $n+1$ times it will give the inequality

$$\left|\frac{L(S_h)-L(S_{nn})}{\alpha}\right|<4h(n+1)MN(b-a). \tag{7}$$

If the surface S_a is taken instead of S, the function $\eta(x, y, \alpha)$ will become

$$\left[\sum_{i=1}^{n}\theta_i(x)-1\right]\omega(x, y, \alpha),$$

which vanishes excepting in the rectangles $(x_i-h<x<x_i+h;\ c<y<d)$, $(i=0, 1, 2, \ldots, n)$. Since the proof of the lemma is not affected by interchanging x and y and replacing a and b by c and d, the inequality

$$\left|\frac{L(S_h)-L(S_a)}{\alpha}\right|<4h(n+1)MN(d-c) \tag{8}$$

is also satisfied for small values of h and $|\alpha|$. Inequalities (7) and (8) can be combined, giving the inequality

$$\left|\frac{L(S_a)-L(S_{nn})}{\alpha}\right|<4h(n+1)MN(b-a+d-c)<4MN\varepsilon+4hMN(b-a+d-c).$$

Since the right member of the last inequality approaches zero with ε, equation (4) may be replaced by the equation

$$\left.\frac{dL(S_a)}{d\alpha}\right|_{\alpha=0}=\int\int_R L'(S;\,x, y)\omega_a(x, y, 0)\,dy\,dx. \tag{9}$$

The assumption that the region R be a rectangle is not essential. If it is any region bounded by a curve of finite length, a suitable rectangle R' can be circumscribed about it, and the function $z(x, y)$ defined arbitrarily and $\omega(x, y, \alpha)$ taken as identically zero over the part of R' not included in R. If the function $L(S)$ is defined in such a way that it depends for its value only on that part of S whose projection on the x, y-plane is R, the derivative $L'(S;\,x, y)$ will vanish everywhere outside of R. The fact that it may be discontinuous at the boundary of R will have no effect on the proof of the above theorem. The theorem states that equation (9) would hold with R' substituted for R, and therefore it holds as it stands.

It will now be proved that if the surface S and the function $L(S)$ satisfy the hypothesis of the last theorem and the surface S minimizes the function $L(S)$, then the derivative $L'(S;\,x, y)$ must vanish at every point of the region

R. Suppose there is a point (x_0, y_0) where the derivative is positive. Then, since it is continuous, there is a finite region including the point (x_0, y_0) where it is also positive. The function $\omega(x, y, \alpha,)$ can be chosen in such a way that its partial derivative $\omega_\alpha(x, y, 0)$ will be positive in this region and zero everywhere else. Then equation (9) will imply that

$$\frac{dL(S_\alpha)}{d\alpha}\bigg|_{\alpha=0} > 0,$$

and the surface S will not minimize the function $L(S)$. If there is a point where the derivative is negative, the argument is essentially the same.

§ 2. *The Derivative of a Double Integral.*

The function $L(S)$ will now be taken as a double integral, and the first necessary condition for a minimum as proved by Lagrange* will be found to be a special case of the last theorem of the preceding section. The arbitrary integer r which designates the order of the approach to the derivative will be taken as 2 in this section.

Let

$$L(S) = \int\int_R f(x, y, z, p, q)\,dydx,$$

where $f(x, y, z, p, q)$ is of class $C^{(2)}$ in the neighborhood of values of the arguments determined by the surface S. The letters p and q represent the partial derivatives of z with respect to x and y. Choose a function $\eta(x, y)$ as in § 1. Then, by definition,

$$L(S_\epsilon) - L(S) = \int_{x_0-\epsilon}^{x_0+\epsilon}\int_{y_0-\epsilon}^{y_0+\epsilon}[f(x, y, z+\eta, p+\eta_x, q+\eta_y) - f(x, y, z, p, q)]\,dydx. \quad (10)$$

If the integrand of this expression is expanded by Taylor's formula, it becomes

$$f_z(x, y, z+\theta\eta, p+\theta\eta_x, q+\theta\eta_y)\eta + f_p\eta_x + f_q\eta_y,$$

where θ is a constant between 0 and 1. It follows from Green's theorem,† and the vanishing of $\eta(x, y)$ along L, the boundary of the region over which the integral is taken, that the equation

$$\int_{x_0-\epsilon}^{x_0+\epsilon}\int_{y_0-\epsilon}^{y_0+\epsilon}\frac{\partial(f_p\eta)}{\partial x}\,dydx = \int_L f_p\eta\,dy = 0$$

is satisfied, and consequently the equation

$$\int_{x_0-\epsilon}^{x_0+\epsilon}\int_{y_0-\epsilon}^{y_0+\epsilon} f_p\eta_x\,dydx = -\int_{x_0-\epsilon}^{x_0+\epsilon}\int_{y_0-\epsilon}^{y_0+\epsilon}\frac{\partial f_p}{\partial x}\eta\,dydx. \quad (11)$$

Similarly

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* Bolza, *loc. cit.*, p. 655.

† Bolza, *loc. cit.*, p. 654.

$$\int_{x_0-\epsilon}^{x_0+\epsilon}\int_{y_0-\epsilon}^{y_0+\epsilon} f_q\eta_y\,dy\,dx = -\int_{x_0-\epsilon}^{x_0+\epsilon}\int_{y_0-\epsilon}^{y_0+\epsilon} \frac{\partial f_q}{\partial y}\eta\,dy\,dx. \qquad (12)$$

If equations (11) and (12) are substituted in the expanded form of equation (10), it becomes

$$L(S_\epsilon)-L(S)=\int_{x_0-\epsilon}^{x_0+\epsilon}\int_{y_0-\epsilon}^{y_0+\epsilon}\left[f_z-\frac{\partial f_p}{\partial x}-\frac{\partial f_q}{\partial y}\right]\eta\,dy\,dx.$$

Since $\eta(x, y)$ has a permanent sign, the mean-value theorem can be applied to the right member of this equation, reducing it to the form

$$L(S_\epsilon)-L(S)=\left[f_z-\frac{\partial f_p}{\partial x}-\frac{\partial f_q}{\partial y}\right]\sigma,$$

where the arguments of the partial derivatives of f are $x_0+\lambda\varepsilon$, $y_0+\mu\varepsilon$; $z(x_0+\lambda\varepsilon, y_0+\mu\varepsilon)+\theta\eta(x_0+\lambda\varepsilon, y_0+\mu\varepsilon)$, $p+\theta\eta_x$, and $q+\theta\eta_y$, and σ is defined by equation (1). The value of the derivative is then seen to be

$$L'(S;\, x_0, y_0)=f_z(x_0, y_0, z(x_0, y_0), p(x_0, y_0), q(x_0, y_0))-\frac{\partial f_p}{\partial x}-\frac{\partial f_q}{\partial y}.$$

It follows from the assumed continuity of the functions $z(x, y)$ and $f(x, y, z, p, q)$ that $L'(S;\, x, y)$ is continuous and approached uniformly with order 2. Therefore the Lagrange equation,

$$f_z-\frac{\partial f_p}{\partial x}-\frac{\partial f_q}{\partial y}=0,$$

is equivalent to the equation

$$L'(S;\, x, y)=0.$$

§ 3. *The Set of Surfaces K.*

In this section there will be given m functions $M_1(S), M_2(S), \dots, M_m(S)$ whose derivatives are continuous and approached uniformly with order r in a neighborhood of order r of the surface S. It will be assumed that there are some points $(x_1, y_1), (x_2, y_2), \dots, (x_m, y_m)$ in the region R which satisfy the inequality

$$\left|\begin{matrix} M'_j(S;\, x_i, y_i) \\ (i, j=1, 2, \dots, m)\end{matrix}\right| \neq 0. \qquad (13)$$

A surface $\bar{S}$ will be said to belong to the set K if it satisfies the equations

$$M_j(\bar{S})-M_j(S)=0, \qquad (j=1, 2, \dots, m).$$

It will now be proved that for every $\varepsilon>0$ there exists a $\delta>0$, which will always be taken as less than or equal to ε, for which the following statement is true: If $\eta(x, y)$ is any function all of whose partial derivatives up to and including those of order r are everywhere less than δ in absolute value, and vanish along the boundary of R, then there is a set of functions $\eta_1(x, y)$, $\eta_2(x, y), \ldots, \eta_m(x, y)$ all of whose partial derivatives, up to and including those of order r, are everywhere less than ε in absolute value, such that $\eta_i(x, y)$ vanishes identically excepting in the region $(x_i-\varepsilon<x<x_i+\varepsilon;\ y_i-\varepsilon<y<y_i+\varepsilon)$, where it has a permanent sign, and the surface

$$S_m: \qquad z=z(x, y)+\eta(x, y)+\sum_{i=1}^{m}\eta_i(x, y) \tag{14}$$

belongs to the set K.

A set of functions $\eta_i(x, y)$ can readily be found which vanish excepting in the specified regions where they have a permanent sign, and whose derivatives are numerically less than ε. Then it must merely be proved that there is a set of multipliers $\alpha_1, \alpha_2, \ldots, \alpha_m$, each numerically less than unity, such that the surface

$$S'_m: \qquad z=z(x, y)+\sum_{i=1}^{m}\alpha_i\eta_i(x, y) \tag{15}$$

belongs to the set K. Then the multipliers α_i can be absorbed into the functions $\eta_i(x, y)$ and the surface S'_m written S_m.

The determinant

$$\left|\begin{matrix}\int_{x_i-\varepsilon}^{x_i+\varepsilon}\int_{y_i-\varepsilon}^{y_i+\varepsilon}M'_j(S;\, x, y)\,\eta_i(x, y)\,dydx\\ (i, j=1, 2, \ldots, m)\end{matrix}\right| \tag{16}$$

must be considered. If the mean-value theorem is applied to each element, it can be reduced to the form

$$\prod_{i=1}^{m}\int_{x_i-\varepsilon}^{x_i+\varepsilon}\int_{y_i-\varepsilon}^{y_i+\varepsilon}\eta_i(x, y)\,dydx\left|\begin{matrix}M'_j(S;\, x_{ij}, y_{ij})\\ (i, j=1, 2, \ldots, m)\end{matrix}\right|,$$

where (x_{ij}, y_{ij}) is some point in the proper region. It follows from inequality (13) and the assumed continuity of the derivatives $M'_j(S;\, x, y)$ that for sufficiently small values of ε this determinant can not vanish, and only such values of ε need be considered.

If ε and the functions $\eta_i(x, y)$ are considered as fixed, the expressions $M_j(S'_m)-M_j(S)$ are completely determined by the function $\eta(x, y)$ and the parameters $\alpha_1, \alpha_2, \ldots, \alpha_m$, and may be represented by the equations

$$M_j(S'_m)-M_j(S)=\Phi_j(H;\, \alpha_1, \alpha_2, \ldots, \alpha_m), \qquad (j=1, 2, \ldots, m),$$

where H represents the surface in the x, y, η-space whose equation is

$\eta=\eta(x, y)$. The surface whose equation is $\eta=0$ will be called H_0. The following equations come directly from the principal theorem of § 1:

$$\frac{\partial\phi_j(H; \alpha_1, \alpha_2, \ldots, \alpha_m)}{\partial\alpha_i}=\int_{x_i-\epsilon}^{x_i+\epsilon}\int_{y_i-\epsilon}^{y_i+\epsilon}M'_j(S'_m;\, x, y)\eta_i(x, y)\,dydx, \qquad (17)$$
$$(i, j=1, 2, \ldots, m);$$

and the determinant

$$\left|\begin{array}{c}\dfrac{\partial\phi_j(H_0; 0, 0, \ldots, 0)}{\partial\alpha_i}\\ (i, j=1, 2, \ldots, m)\end{array}\right|$$

is equal to the determinant (16), which can not vanish. The functions $\phi_j(H, \alpha_1, \alpha_2, \ldots, \alpha_m)$ and their partial derivatives given by equations (17) are seen to be continuous in all arguments in the neighborhood of $(H_0; 0, 0, \ldots, 0)$. Thus the hypothesis of an existence theorem in the paper entitled "A Generalization of Volterra's Derivative of a Function of a Curve,"* mentioned in the introduction, is satisfied. The fact that H now represents a surface instead of a curve has no effect on the proof of this theorem. The conclusion of this existence theorem is that for every $\varepsilon>0$ sufficiently small there is a $\delta>0$ such that to every admissible function $\eta(x, y)$ there corresponds a unique set of constants $\alpha_1, \alpha_2, \ldots, \alpha_m$, each numerically less than unity, which satisfy the equations

$$\phi_j(H; \alpha_1, \alpha_2, \ldots, \alpha_m)=0, \qquad (j=1, 2, \ldots, m).$$

The surface S'_m then belongs to the set K. It is evident that if a δ is effective for a particular value of ε, as ε_0, it is also effective for every value of ε greater than ε_0, since the functions $\eta_i(x, y)$ can be taken identically zero over as much of the regions $(x_i-\varepsilon<x<x_i+\varepsilon;\ y_i-\varepsilon<y<y_i+\varepsilon)$ as is desired, provided they are different from zero in a part of each.

§ 4. *The Derivative Relative to the Set K.*

The derivative $L'(S; x, y; x_1, y_1, \ldots, x_m, y_m)$ of the function $L(S)$ relative to the set of surfaces K can now be defined, and evaluated in terms of the derivatives $L'(S; x, y)$, $M'_1(S; x, y)$, ..., $M'_m(S; x, y)$, if they are continuous and approached uniformly with any finite order r.

The function $\eta(x, y)$ discussed in the preceding section will be restricted by the additional condition that it vanish identically excepting in the region $(x_0-\varepsilon<x<x_0+\varepsilon;\ y_0-\varepsilon<y<y_0+\varepsilon)$, where it has a permanent sign and is not identically zero. *Then the limit*

---

* Fischer, *loc. cit.*, p. 373.

$$L'(S;\, x_0,\, y_0;\, x_1,\, y_1,\, \ldots,\, x_m,\, y_m) = \lim_{\epsilon=0} \frac{L(S_m) - L(S)}{\sigma}$$

will be called the derivative of the function $L(S)$ *at the point* (x_0, y_0), *relative to the set of surfaces* K *and the points* (x_1, y_1), (x_2, y_2), $\ldots$, (x_m, y_m). The integral σ and the surface S_m have already been defined by equations (1) and (14).

This derivative can be evaluated as follows: Define the surfaces S_0, S_1, $\ldots$, S_{m-1} by the equations

$$S_0{:}\qquad z = z(x, y) + \eta(x, y),$$

$$S_i{:}\qquad z = z(x, y) + \eta(x, y) + \sum_{k=1}^{i} \eta_k(x, y), \qquad (i = 1, 2, \ldots, m-1);$$

and for convenience in the notation let

$$\sigma_i = \int_{x_i-\epsilon}^{x_i+\epsilon} \int_{y_i-\epsilon}^{y_i+\epsilon} \eta_i(x, y)\, dy dx, \qquad (i = 1, 2, \ldots, m).$$

Then the derivative may be written in the form

$$L'(S;\, x_0,\, y_0;\, x_1,\, y_1,\, \ldots,\, x_m,\, y_m) = \\ \lim_{\epsilon=0} \left[\frac{L(S_0) - L(S)}{\sigma} + \sum_{i=1}^{m} \frac{L(S_i) - L(S_{i-1})}{\sigma_i} \cdot \frac{\sigma_i}{\sigma} \right],$$

which is easily reduced to the equation

$$L'(S;\, x_0,\, y_0;\, x_1,\, y_1,\, \ldots,\, x_m,\, y_m) = \\ L'(S;\, x_0,\, y_0) + \sum_{i=1}^{m} L'(S;\, x_i,\, y_i) \lim_{\epsilon=0} \frac{\sigma_i}{\sigma}. \qquad (18)$$

It is now necessary to evaluate the expressions

$$\lim_{\epsilon=0} \frac{\sigma_i}{\sigma}.$$

Since S_m belongs to the set K, the equations

$$\frac{M_j(S_m) - M_j(S)}{\sigma} = 0, \qquad (j = 1, 2, \ldots, m),$$

are always satisfied. If ϵ is made to approach zero, they become

$$M_j'(S;\, x_0,\, y_0) + \sum_{i=1}^{m} M_j'(S;\, x_i,\, y_i) \lim_{\epsilon=0} \frac{\sigma_i}{\sigma} = 0, \qquad (j = 1, 2, \ldots, m).$$

The determinant of the coefficients of $\lim \sigma_i/\sigma$ is not zero, and therefore the solution is unique. Solving for these limits and substituting their values in equation (18), it becomes

$$L'(S;x_0,y_0;x_1,y_1,\ldots,x_m,y_m)=L'(S;x_0,y_0)-$$
$$\sum_{k=1}^{m}L'(S;x_k,y_k)\frac{\begin{vmatrix} M'_1(S;x_1,y_1)\,..\,M'_1(S;x_{k-1},y_{k-1})M'_1(S;x_0,y_0)\,..\,M'_1(S;x_m,y_m)\\ M'_2(S;x_1,y_1)\,..\,M'_2(S;x_{k-1},y_{k-1})M'_2(S;x_0,y_0)\,..\,M'_2(S;x_m,y_m)\\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ M'_m(S;x_1,y_1)\,\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\,M'_m(S;x_m,y_m)\end{vmatrix}}{\begin{vmatrix} M'_j(S;x_i,y_i)\\ (i,j=1,2,\ldots,m)\end{vmatrix}}.$$

This may be written either in the form

$$L'(S;x_0,y_0;x_1,y_1,\ldots,x_m,y_m)=L'(S;x_0,y_0)+\sum_{k=1}^{m}\lambda_k M'_k(S;x_0,y_0),\qquad(19)$$

where the quantities $\lambda_1,\lambda_2,\ldots,\lambda_m$ are defined by the equations

$$\lambda_k=(-1)^k\frac{\begin{vmatrix} L'(S;x_1,y_1) & L'(S;x_2,y_2)\ldots & L'(S;x_m,y_m)\\ M'_1(S;x_1,y_1) & M'_1(S;x_2,y_2)\ldots & M'_1(S;x_m,y_m)\\ \cdots\cdots & \cdots\cdots & \cdots\cdots\\ M'_{k-1}(S;x_1,y_1) & M'_{k-1}(S;x_2,y_2)\ldots & M'_{k-1}(S;x_m,y_m)\\ M'_{k+1}(S;x_1,y_1) & M'_{k+1}(S;x_2,y_2)\ldots & M'_{k+1}(S;x_m,y_m)\\ \cdots\cdots & \cdots\cdots & \cdots\cdots\\ M'_m(S;x_1,y_1) & M'_m(S;x_2,y_2)\ldots & M'_m(S;x_m,y_m)\end{vmatrix}}{\begin{vmatrix} M'_j(S;x_i,y_i)\\ (i,j=1,2,\ldots,m)\end{vmatrix}},\qquad(20)$$

or in the form

$$\begin{vmatrix} M'_j(S;x_i,y_i)\\ (i,j=1,2,\ldots,m)\end{vmatrix}L'(S;x_0,y_0;x_1,y_1,\ldots,x_m,y_m)=$$
$$\begin{vmatrix} L'(S;x_0,y_0) & L'(S;x_1,y_1)\ldots & L'(S;x_m,y_m)\\ M'_1(S;x_0,y_0) & M'_1(S;x_1,y_1)\ldots & M'_1(S;x_m,y_m)\\ M'_2(S;x_0,y_0) & M'_2(S;x_1,y_1)\ldots & M'_2(S;x_m,y_m)\\ \cdots\cdots & \cdots\cdots & \cdots\cdots\\ M'_m(S;x_0,y_0) & M'_m(S;x_1,y_1)\ldots & M'_m(S;x_m,y_m)\end{vmatrix}.\qquad(21)$$

Since $L'(S;x,y)$, $M'_1(S;x,y)$, $\ldots$, $M'_m(S;x,y)$ are continuous and approached uniformly with order r, $L'(S;x,y;x_1,y_1,\ldots,x_m,y_m)$ is also.

The following equations, which will be used later, follow immediately from equation (21):

$$L'(S;x_i,y_i;x_1,y_1,\ldots,x_m,y_m)=0,\qquad(i=1,2,\ldots,m).$$

The extension of Volterra's theorem proved in § 1 can readily be proved for the case where only surfaces of the set K are considered. Suppose that the one-parameter family of surfaces

$$S_a: \qquad z=z(x, y)+\omega(x, y, \alpha)$$

discussed in § 1 belongs to the set K. Then it follows from equation (9) that

$$\left.\frac{dL(S_a)}{d\alpha}\right|_{a=0}=\int\int_R L'(S;\ x, y)\omega_a(x, y, 0)\,dydx, \tag{22}$$

and similarly

$$\left.\frac{dM_j(S_a)}{d\alpha}\right|_{a=0}=\int\int_R M'_j(S;\ x, y)\omega_a(x, y, 0)\,dydx, \qquad (j=1, 2, \ldots, m). \tag{23}$$

The left members of equation (23) must vanish since S_a belongs to the set K. If the right members are multiplied by $\lambda_1, \lambda_2, \ldots, \lambda m$ respectively, equated to zero, and added to the right member of equation (22), it becomes

$$\left.\frac{dL(S_a)}{d\alpha}\right|_{a=0}=\int\int_R L'(S;\ x, y;\ x_1, y_1, \ldots, x_m, y_m)\omega_a(x, y, 0)\,dydx. \tag{24}$$

It will now be proved that if the surface S minimizes the function $L(S)$ relative to the set of surfaces K, then the derivative $L'(S;\ x, y;\ x_1, y_1, \ldots, x_m, y_m)$ must vanish for all values of x and y in the region R.

Suppose that there is a point (x_0, y_0) where it does not vanish, and to fix ideas, suppose that it is positive at (x_0, y_0). Then, since the derivative is continuous, two positive constants h and k can be found such that the inequality

$$L'(S;\ x, y;\ x_1, y_1, \ldots, x_m, y_m)>k \tag{25}$$

is satisfied at every point in the region $(x_0-h\leqq x\leqq x_0+h;\ y_0-h\leqq y\leqq y_0+h)$. Since the derivative vanishes at each of the points $(x_1, y_1), (x_2, y_2,), \ldots, (x_m, y_m)$, for every $\delta>0$ there exists an $\varepsilon>0$ such that the inequality

$$|L'(S;x, y;x_1, y_1, \ldots, x_m, y_m)|<\delta \tag{26}$$

is satisfied at every point in each of the regions

$$(x_i-\varepsilon\leqq x\leqq x_i+\varepsilon;\ y_i-\varepsilon\leqq y\leqq y_i+\varepsilon), \qquad (i=1, 2, \ldots, m).$$

Select a function $\eta(x, y)$ which vanishes at all points outside of the region $(x_0-h<x<x_0+h;\ y_0-h<y<y_0+h)$, where it is positive, and m functions $\eta_i(x, y)$ which vanish at all points outside of the respective regions $(x_i-\varepsilon< x<x_i+\varepsilon;\ y_i-\varepsilon<y<y_i+\varepsilon)$, where they are positive. Then it follows from the proof of the existence theorem of § 3, that if ε is taken sufficiently small, for every value of α numerically less than a fixed positive number, there exists a unique set of numbers $\alpha_1, \alpha_2, \ldots, \alpha_m$, each numerically less than another fixed number, such that the surface

$$S_a: \qquad z=z(x, y)+\alpha\eta(x, y)+\sum_{i=1}^{m}\alpha_i\eta_i(x, y)$$

belongs to the set K. The function $\omega(x, y, \alpha)$ will now be defined by the equation

$$\omega(x, y, \alpha) = \alpha\eta(x, y) + \sum_{i=1}^{m} \alpha_i \eta_i(x, y). \tag{27}$$

As the existence of the partial derivative $\omega_\alpha(x, y, 0)$ can not be easily proved, the assumption that it exists will be replaced by the less exacting assumption that there is a constant N, such that the inequality

$$\left| \frac{\omega(x, y, \alpha)}{\alpha} \right| < N \tag{28}$$

is satisfied, for all numerically small values of α. This necessitates some changes in the statement and proof of the extension of Volterra's theorem. Equation (4) must be replaced by the equation

$$\lim_{\epsilon,\, \alpha=0} \left[\frac{L(S_{nn}) - L(S)}{\alpha} - \int_a^b \int_c^d L'(S;\, x, y) \frac{\omega(x, y, \alpha)}{\alpha}\, dy dx \right] = 0,$$

and equation (9) by the equation

$$\lim_{\alpha=0} \left[\frac{L(S_\alpha) - L(S)}{\alpha} - \iint_R L'(S;\, x, y) \frac{\omega(x, y, \alpha)}{\alpha}\, dy dx \right] = 0.$$

The proof of the theorem will then be valid. In the present section equations (22) and (23) must be replaced by the equations

$$\lim_{\alpha=0} \left[\frac{L(S_\alpha) - L(S)}{\alpha} - \iint_R L'(S;\, x, y) \frac{\omega(x, y, \alpha)}{\alpha}\, dy dx \right] = 0, \tag{29}$$

and

$$\lim_{\alpha=0} \left[\frac{M_j(S_\alpha) - M_j(S)}{\alpha} - \iint_R M_j'(S;\, x, y) \frac{\omega(x, y, \alpha)}{\alpha}\, dy dx \right] = 0, \tag{30}$$

$$(j = 1, 2, \ldots, m).$$

Since the surface S_α belongs to the set K, equations (30) may be written

$$\lim_{\alpha=0} \iint_R M_j'(S;\, x, y) \frac{\omega(x, y, \alpha)}{\alpha}\, dy dx = 0, \qquad (j = 1, 2, \ldots, m). \tag{31}$$

If equations (31) are multiplied by $\lambda_1, \lambda_2, \ldots, \lambda_m$, respectively, and subtracted from equation (29), it becomes by virtue of equation (19)

$$\lim_{\alpha=0} \left[\frac{L(S_\alpha) - L(S)}{\alpha} - \iint_R L'(S;\, x, y;\, x_1, y_1, \ldots, x_m, y_m) \frac{\omega(x, y, \alpha)}{\alpha}\, dy dx \right] = 0, \tag{32}$$

which is the desired generalization of equation (24).

It will next be proved that if the function $\omega(x, y, \alpha)$ is defined by equation (27) a constant N must exist, satisfying inequality (28). Let

$$
\begin{aligned}
S_0 &: \quad z=z(x, y)+\alpha\eta(x, y), \\
S_i &: \quad z=z(x, y)+\alpha\eta(x, y)+\sum_{k=1}^{i}\alpha_k\eta_k(x, y), \qquad (i=1, 2, \ldots, m).
\end{aligned}
$$

Then, since S_m, which is the same as S_α, belongs to the set K, the equations

$$\frac{M_j(S_m)-M_j(S)}{\alpha}=\frac{M_j(S_0)-M_j(S)}{\alpha}+\sum_{i=1}^{m}\frac{M_j(S_i)-M_j(S_{i-1})}{\sigma_i}\cdot\frac{\sigma_i}{\alpha}=0$$

are satisfied, where $\sigma_1, \sigma_2, \ldots, \sigma_m$ are defined by the equations

$$\sigma_i=\int_{x_i-\epsilon}^{x_i+\epsilon}\int_{y_i-\epsilon}^{y_i+\epsilon}\alpha_i\eta_i(x, y)\,dydx.$$

When ε and α both approach zero, the determinant of the coefficients of σ_i/α approaches the determinant (13) uniformly, and therefore it can not vanish if $|\alpha|$ and ε are sufficiently small. The terms $1/\alpha[M_j(S_0)-M_j(S)]$ approach finite limits as α approaches zero. Thus a constant N', independent of ε and α, can be found, such that, if ε is given any value less than a fixed positive constant and then α is taken less numerically than another constant, then the inequalities $|\sigma_i/\alpha|<N'$ will be satisfied. Since the expressions

$$\frac{\sigma_i}{\alpha_i}=\int_{x_i-\epsilon}^{x_i+\epsilon}\int_{y_i-\epsilon}^{y_i+\epsilon}\eta_i(x, y)\,dydx$$

are positive and independent of α, the ratios α_i/α do not become infinite as α approaches zero; and if $\omega(x, y, \alpha)$ is defined by equation (27), a constant N can be found satisfying equation (28). Since N is not independent of ε, it will be better to use N' in what follows.

Equation (32) may now be written

$$
\begin{aligned}
\frac{L(S_\alpha)-L(S)}{\alpha}=&\int_{x_0-h}^{x_0+h}\int_{y_0-h}^{y_0+h}L'(S; x, y; x_1, y_1, \ldots, x_m, y_m)\eta\,dydx \\
&+\sum_{i=1}^{m}\int_{x_i-\epsilon}^{x_i+\epsilon}\int_{y_i-\epsilon}^{y_i+\epsilon}L'(S; x, y; x_1, y_1, \ldots, x_m, y_m)\frac{\alpha_i\eta_i}{\alpha}\,dydx+\xi(\alpha),
\end{aligned}
$$

where $\xi(\alpha)$ approaches zero with α. It follows from this equation and inequalities (25) and (26) that the inequality

$$\frac{L(S_\alpha)-L(S)}{\alpha}>k\int_{x_0-h}^{x_0+h}\int_{y_0-h}^{y_0+h}\eta\,dydx-mN'\delta-|\xi(\alpha)|$$

is satisfied. Since the first term of the right member is positive and independent of δ, ε, and α, if δ is taken sufficiently small the inequality

$$\frac{L(S_\alpha)-L(S)}{\alpha}>0$$

is satisfied for small values of α both positive and negative. Therefore the surface S does not minimize the function $L(S)$. If $L'(S; x_0, y_0; x_1, y_1, \ldots, x_m, y_m)$ were negative, the argument would be essentially the same.

The quantities $\lambda_1, \lambda_2, \ldots, \lambda_m$ defined by equations (20) are functions of the $2m$ arguments $x_1, y_1, \ldots, x_m, y_m$; but in the special case where the derivative vanishes identically, they are constants, as can be proved as follows: Let $x'_1, y'_1, \ldots, x'_m, y'_m$ be a fixed set of values for $x_1, y_1, \ldots, x_m, y_m$, and let those letters without accents be considered as variables. Then the equations

$$\begin{aligned} &L'(S; x'_i, y'_i; x_1, y_1, \ldots, x_m, y_m) = \\ &L'(S; x'_i, y'_i) + \sum_{j=1}^{m} \lambda_j(x_1, y_1, \ldots, x_m, y_m) M'_j(S; x'_i, y'_i) = 0, \\ &\qquad\qquad\qquad\qquad\qquad\qquad (i=1, 2, \ldots, m), \end{aligned}$$

are satisfied identically. Comparing the solutions of these equations with equations (20), it is evident that

$$\lambda_j(x_1, y_1, \ldots, x_m, y_m) = \lambda_j(x'_1, y'_1, \ldots, x'_m, y'_m), \qquad (j=1, 2, \ldots, m).$$

That is, $\lambda_1, \lambda_2, \ldots, \lambda_m$ are constants.

§ 5. *The Derivative of a Double Integral Relative to the Set K.*

If the functions $L(S)$, $M_1(S)$, $\ldots$, $M_m(S)$ are defined by the equations

$$L(S) = \iint_R f(x, y, z, p, q)\, dydx, \tag{33}$$

$$M_j(S) = \iint_R g_j(x, y, z, p, q)\, dydx, \qquad (j=1, 2, \ldots, m), \tag{34}$$

where $z(x, y)$, $f(x, y, z, p, q)$ and $g_j(x, y, z, p, q)$ are of class $C^{(2)}$, and the determinant (13) is distinct from zero, the derivative of $L(S)$ with respect to the restricted set of surfaces is given by the equation

$$L'(S; x, y; x_1, y_1, \ldots, x_m, y_m) = f_z - \frac{\partial f_p}{\partial x} - \frac{\partial f_q}{\partial y} + \sum_{j=1}^{m} \lambda_j \left[g_{jz} - \frac{\partial g_{jp}}{\partial x} - \frac{\partial g_{jq}}{\partial y} \right], \tag{35}$$

where the arguments of the partial derivatives of f and g_j are x, y, $z(x, y)$, $p(x, y)$, $q(x, y)$, and the coefficients $\lambda_1, \lambda_2, \ldots, \lambda_m$ are defined by the equations

$$\lambda_k = (-1)^k \frac{\begin{vmatrix} \left[f_z - \frac{\partial f_p}{\partial x} - \frac{\partial f_q}{\partial y}\right]_{x_1, y_1} & \cdots\cdots\cdots & \left[f_z - \frac{\partial f_p}{\partial x} - \frac{\partial f_q}{\partial y}\right]_{x_m, y_m} \\ \left[g_{1z} - \frac{\partial g_{1p}}{\partial x} - \frac{\partial g_{1q}}{\partial y}\right]_{x_1, y_1} & \cdots\cdots\cdots & \left[g_{1z} - \frac{\partial g_{1p}}{\partial x} - \frac{\partial g_{1q}}{\partial y}\right]_{x_m, y_m} \\ \cdots\cdots\cdots & \cdots\cdots\cdots & \cdots\cdots\cdots \\ \left[g_{k-1z} - \frac{\partial g_{k-1p}}{\partial x} - \frac{\partial g_{k-1q}}{\partial y}\right]_{x_1, y_1} & \cdots & \left[g_{k-1z} - \frac{\partial g_{k-1p}}{\partial x} - \frac{\partial g_{k-1q}}{\partial y}\right]_{x_m, y_m} \\ \left[g_{k+1z} - \frac{\partial g_{k+1p}}{\partial x} - \frac{\partial g_{k+1q}}{\partial y}\right]_{x_1, y_1} & \cdots & \left[g_{k+1z} - \frac{\partial g_{k+1p}}{\partial x} - \frac{\partial g_{k+1q}}{\partial y}\right]_{x_m, y_m} \\ \cdots\cdots\cdots & \cdots\cdots\cdots & \cdots\cdots\cdots \\ \left[g_{mz} - \frac{\partial g_{mp}}{\partial x} - \frac{\partial g_{mq}}{\partial y}\right]_{x_1, y_1} & \cdots\cdots\cdots & \left[g_{mz} - \frac{\partial g_{mp}}{\partial x} - \frac{\partial g_{mq}}{\partial y}\right]_{x_m, y_m} \end{vmatrix}}{\begin{vmatrix} \left[g_{jz} - \frac{\partial g_{jp}}{\partial x} - \frac{\partial g_{jq}}{\partial y}\right]_{x_i, y_i} \\ (i, j = 1, 2, \ldots, m) \end{vmatrix}}$$

The right member of equation (35) equated to zero constitutes the first necessary condition that the surface S minimize the double integral (33) relative to the surfaces which give fixed values to the integrals (34).

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